LOGIC AND MATHEMATICS

AN OPINIONATED HISTORY OF A TROUBLED RELATIONSHIP

Mike Arnautov, 2009

I was recently asked: “Does logic underpin all of mathematics?” It was one of those innocent questions to which you open your mouth to reply... and no coherent reply comes out. The truth is, there is no easy answer, and in the end I was strongly tempted to resort to the universal philosophical brush-off: “Define ‘logic’ and define ‘underpin’ -- I’ll let you off defining ‘does’ and ‘all of mathematics’” But I am not that mean. What follows is my somewhat spruced-up eventual attempt to give an honest and reasonably meaningful answer. It is not meant to be a rigorous philosophical exposition. The idea is to give a fairly simple, and yes, of necessity somewhat opinionated, overview of the complexities involved.

For a long time (ever since Aristotle’s list of logical figures) logic existed as a rather curious, semi-detached adjunct of mathematics. It was useful in learned disputations, but not seriously of any practical use in serious mathematical work. This is not to belittle Aristotle’s achievement! With the benefits of hindsight we can see the extent to which his explorations of logic anticipated much of the modern approach to the subject. However, for well over 2000 years mathematicians had nothing much to say about logic, which was believed to be fully accounted for by Aristotle’s work. Mathematically interesting topics were to be found elsewhere – e.g. in solving polynomial equations, calculating areas and volumes of various geometric objects, or uncovering interesting geometric relationships – and the requisite proofs were done by self-evident demonstration, extended where necessary by common sense. While such an approach works fine up to a point, there are issues.

Firstly, what is self-evidently demonstrated for somebody like Archimedes, is probably not so for most of the rest of us[1]. And the common-sense logic is also somewhat peculiar in that it is non-monotonic. This means that its reasoning is defeasible. Any conclusion one arrives at by common sense, may be negated by further evidence - always so in principle, often so in practice. (Birds fly; but ostriches do not). Clearly, that is not a very satisfactory situation where mathematical proofs are concerned. A mathematical proof is expected to be valid regardless of any subsequent developments. We are safe in the knowledge that the Pythagoras’s Theorem will be presented as valid to geometry students for as long as Euclidean geometry continues to be taught.[2]

Formal mathematical notation helps, but we are so used to it that not many people realise how recent it actually is. And it got developed in a completely pragmatic, ad hoc manner, driven by (of course) the requirements of self-evident demonstration and common sense. Even as late as the last decade of the 16th century the French mathematician François Viète found it natural to write a general quadratic equation as “B in A Quadratum, Plus D plano in A, aequar Z solido”[3], where we would write “$ax^2 + bx + c = 0$”.

---

1 Recent fascinating discoveries documented in The *Archimedes Codex* by Reviel Netz and William Noel, strongly suggest that we have been seriously underestimating the sophistication of Greek mathematics.

2 Philosophers may be tempted to conclude that the Pythagoras Theorem and other such mathematical results are true ‘in all possible worlds’, but personally I think that’s a bit of an enthusiastic over-generalisation. The nature of mathematical truths is a tricky enough subject, and adding the ill-defined concept of ‘all possible worlds’ only serves to muddy the waters.

3 Quoted by John Derbyshire in his excellent book *Unknown Quantity, a Real and Imaginary History of Algebra.*

Things came to a head with the invention of the calculus (either variety - Newton's or Leibniz's), because its originators treated mathematical reasoning in a rather cavalier way, which did not go down too well with all mathematicians. To be fair to Newton and Leibniz, they were right in their constructions, even if their mathematical 'health and safety' record was lamentable from the modern point of view. And it must be acknowledged that Euler – one of the greatest mathematicians ever - had a similar disregard for mathematical belt-and-braces, but was brilliant enough to manage quite well without, thank you. Therein perhaps lies the difference between genius and mere talent.

A great deal of mathematical effort in the second half of the 19th century went into underpinning of the calculus in some rigorous way, culminating in the rigorous 'epsilon/delta' reformulation by Karl Weierstrass . At about the same time Boole was busy algebraising logic, and his work finally brought logic properly into the fold of mathematics, allowing logical propositions to be written down as mathematical equations. This remarkable achievement made it quite natural to ask: given that there was now a proper mathematical logic available, might it be possible to harness it as a replacement for self-evident demonstrations and common-sense in mathematical proofs? In other words, might not all mathematics be in fact reducible to logic? This possibility looked so plausible that at one stage a complete takeover of mathematics by logicians seemed quite imminent. While the prospect was causing a stir within the mathematical community, philosophers probably looked on in benign amusement. From their perspective, such a development would have been simply a vindication of Hume's view of mathematics as a vastly extended tautology.

Gottlob Frege was the first to provide what looked like exactly the sort of system that was required, and as it happens, his work also greatly clarified
Weiers/home/mla/adv/prep/www/acode/acode-tutorial-2025-Jul-02.epub
trass's formalisation of the calculus. But all
imperial ambitions come to grief sooner or later. In this particular case, it happened all too soon, As is well known, along came Bertrand Russell and poked a large hole in Frege's construction by pointing out the paradoxical nature of a set of all sets not containing themselves. Russell himself helped to patch the system with his theory of types, but that was very visibly an ad hoc solution. Other approaches have been suggested since then, but there does not seem to be an obviously natural fix.

For a while it seemed that formalists led by David Hilbert could get around the problem by reducing all mathematics to a study of axiomatic systems. One defines some arbitrary axioms, one defines some arbitrary but strictly formal rules for combining and transforming statements, and then one explores the system of statements that grows out of those axioms on the basis of those rules. The only requirement is that of consistency – it must not be possible to arrive at incompatible conclusions by two separate and separately valid chains of transformations.

Even that approach went down in flames when Kurt Gödel pulled the undecidability rabbit out of his hat. He showed quite conclusively, that regardless of the system of axioms and transformation rules one started with, there always remained true mathematical statements, which were not logically deducible from those axiomatic premises. This amazing feat of deduction put logic firmly in its place: mathematics subsumes formal logic, not the other way around.

However, the formalist programme had some interesting consequences. For example, it became quite clear that Frege's predicate calculus was but one of a (possibly unlimited number) of 'systems of logic', and other such systems were explored, nothing being sacred. You don't like the excluded middle? Here's ternary logic for you! Sir doesn't fancy the monotonicity of the predicate calculus? Have a non-monotonic logic instead! Determinism doesn't suit? We have a probabilistic logic just for you, madam!

With logicians in such merry and fruitful disarray, things became even jollier when the principle of self-evident demonstration in mathematical proofs got pushed to its breaking limit by the final proof of the notorious four-colour-map conjecture in 1976 by Kenneth Appel and Wolfgang Haken. The necessary demonstration was provided by a large computer program examining thousands of possible cases, and a number of mathematicians objected rather strongly that this did not constitute a proper proof. This particular methodological controversy remains unresolved, though the opposition to such demonstrations seem to have subsided with the passing years.

Taking a more general look at the history of the subject, another problem becomes apparent, which paradoxically enough seems also to be lurking at the heart of the formalist programme with its arbitrary choices of axiomatic systems. If one considers the way mathematics develops, it is clear that sometimes it does so by creative leaps, not derivable from logic. For example, the abstraction of fields, rings, ideals etc. from the humble basis of arithmetic was ‘under-determined’, i.e. in no way dictated by ‘logic’ – it just made sense in some hard-to-pin-down, yet obvious manner; and most importantly, it posed new, interesting questions for mathematicians to ponder.

And yet, despite all of the above, few mathematicians would deny that there is something like an inherent structure underlying all of mathematics. Sudden convergence of (e.g.) the group theory and the number theory via the discovery and exploration of the Monster Group[4] strongly suggests an objective underlying unity. But that’s the most we can say at present about the ‘real logic’ of mathematics. So perhaps my original knee-jerk reaction of demanding a definition of ‘logic’ supposed to underpin mathematics was not so unreasonable after all.

Perplexed by the above, the reader may quite sensibly wish to know what is currently used in mathematical practice. To the best of my knowledge it is a mixture of formalism and self-evident demonstration, seasoned by common sense.

So there you are. That’s my complex answer to the seemingly simple question whether logic underpins all of mathematics.

- o O o -

4 Documented in recent popular books by Marcus du Sautoy (*Finding Moonshine*) and Ian Stewart (*Why Beauty Is Truth*).